5 The meaning of spacetime singularities

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Abstract A recent extension (not modification) of semi-Riemannian geometry and general relativity has been proven to work for a large class of singularities, which includes the major known ones, and to provide a description of them in terms of finite invariant geometric and physical objects. An interesting consequence is that one can no longer conclude that the existence of these singularities represents the breakdown of general relativity. After a brief review of these results, some implications on the nature and ontology of spacetime are discussed, in particular at singularities. The proposed approach suggests that it is relevant to understand what geometric and physical objects are more fundamental and why. In order to achieve this goal, the underlying mathematical structures of spacetime are deconstructed. In particular, the very notion of metric, connection, curvature, causal structure, stress-energy tensor, are revised. The analysis suggests that the structure of lightcones plays a more fundamental role than other structures.

1 Introduction

In this article I will discuss the geometric and physical interpretation of spacetime singularities occurring in *general relativity*.

General relativity has two main problems: the prediction of *singularities* [22], and the problem of *quantization*. Despite these problems, the predictions of general relativity continue to be confirmed by experiment, culminating recently with the detection of gravitational waves resulting from the merging of two black holes [18].

Given the repeated experimental confirmation of the predictions of general relativity as compared to the alternative theories, we should consider more carefully what general relativity itself has to say about singularities. This motivated my research program to find a natural formulation of general relativity in terms of variables that remain finite at singularities, *cf.* [33] and references therein. I will briefly review these results, and then discuss the ge-

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ometric and physical interpretation of singularities and singular spacetimes, as well as the implications on the nature and ontology of spacetime.

2 Spacetime singularities

In general relativity the metric tensor is dynamic, and so are the Levi-Civita connection (needed for covariant derivatives), the geodesics, and the Riemann curvature. Einstein's equation specifies how the geometry and the matter fields evolve in an interdependent way. The metric is a symmetric tensor, specified at each point, in a coordinate system, by a symmetric 4×4 matrix. The entries g_{ab} of the matrix are dynamical quantities. Both geometry and the field equations are based on the metric, and they can work in their usual form only if we assume that the determinant of g_{ab} never vanishes, and that each component g_{ab} is finite, and that their first and second order partial derivatives are finite too. However, there is no guarantee that this always happens. The dynamics changes the components g_{ab} of the metric sometimes violently, and there is no way to be sure that the determinant will never vanish, and that none of its components will never become infinite. When any of these two possibilities happens, the metric is *singular*.

The first exact solutions of Einstein's equation showed that this may happen. Both the Schwarzschild black hole solution [26, 25] and the big-bang cosmological model of Friedmann-Lemaître-Robertson-Walker (FLRW) [10, 11, 17] have singularities. While this was hoped to be a special case due to the too high degree of symmetry of the solutions, it was proven, through the *singularity theorems* by Penrose [22, 23] and Hawking [12, 13, 14, 15], that it is actually much more general, and unavoidable under reasonable conditions.

The problem with singularities is that the metric becomes singular. This means that some of the metric tensor components g_{ab} or g^{ab} become infinite. This prevents the construction of the covariant derivative $\Gamma^a{}_{bc}$ (since $\Gamma^a{}_{bc}$ requires the inverse of the metric) and the Riemann curvature $R^a{}_{bcd}$. However, in [37] I show that differential geometry can be extended in a natural and invariant way to singular metrics g_{ab} which are smooth and become degenerate ($\det g = 0$).

3 Does spacetime break down at singularities?

Is the prediction of singularities the omen of the breakdown of general relativity? I will explain that it is not the case, and that only the usual equations we use to understand them have this problem. But it is possible to change the equations, not by modifying them, but in a way similar to a change of variables. The resulting equations are expressed in terms of variables that do not blow up at singularities, and outside the singularities the solutions coincide with the standard ones. The new variables are as natural as the usual ones and from certain points of view more fundamental, both geometrically and physically.

In [37, 43, 34] I studied metric singularities for which the components g_{ab} remain finite, and the determinant of the metric vanishes. Let us call such singularities *benign*, and let us call *malign singularities* those for which g_{ab} blows up for some components. The result was a generalization of semi-Riemannian geometry, which I applied in subsequent articles to the spacetime singularities in general relativity (see [33] and references therein). For a large class of benign singularities, one can still have geometric and physical objects that remain finite at the singularities, which satisfy invariant field equations equivalent to the usual ones outside the singularities.

Even for the benign singularities, those for which the metric is smooth but the determinant vanishes, one cannot construct the covariant derivative and the Riemann curvature as usually. The reason is that $\Gamma^{a}{}_{bc}$ and $R^{a}{}_{bcd}$ are constructed using the inverse of the metric, g^{ab}, which blows up when $\det g_{ab} = 0$.

But the *lower covariant derivative* (which can be expressed in terms of the *Christoffel symbols* of the first kind Γ_{abc}) and the lower-index form of the Riemann curvature R_{abcd} remain finite at such singularities. This turned out to be enough to describe a large class of singularities, and to rewrite Einstein's equation in terms of quantities that remain finite, and the solutions are still equivalent to the solutions of the original Einstein equation outside the singularities [37, 35]. This applies to FLRW and more general big bang solutions [41, 30, 32].

Although for the malign singularities the problem is a bit more difficult, there is a solution for their case as well. All examples of stationary black holes contain malign singularities, but they can be reduced to benign singularities by using singular coordinate transformations, similarly to the case of the event horizon, which was resolved by Eddington [8] and Finkelstein [9]. Of course, unlike the case of the event horizon, for which there are singular coordinate transformations that remove the singularity, for the $r = 0$ singularities this will not work to remove it, because the *Kretschmann invariant* $R_{abcd}R^{abcd}$ blows up. But the singularity can be made benign by such transformations, and then the mathematical apparatus I developed in [37, 43, 34] can be applied. In the following I will detail this approach.

The fact that black hole singularities are apparently not benign, but malign, may be explained by considering that the usual coordinates are themselves singular, similarly to the case of the event horizon, which was resolved by Eddington and Finkelstein. The singular coordinate transformation I used, similar to their method, could be used to make the $r = 0$ singularity of black holes smooth, albeit degenerate [31, 29, 40]. The mentioned methods developed for degenerate metrics with benign singularities can then be applied, and the Schwarzschild solution can be extended analytically beyond the singularity. Singularities turn out to be compatible with global hyperbolicity [31, 38], allowing thus the conservation of information during black hole evaporation.

Not only that the singularities in general relativity turned out to be understandable in terms of finite quantities, but they may also provide a solution to the problem of quantum gravity. The geometric understanding of singularities from this approach leads to the conclusion that they are accompanied by *dimensional reduction* effects, which are researched in the last years because they allow the removal of infinities in *perturbative quantum gravity*. Many, perhaps most of the approaches to quantum gravity have something in common – they either imply, or rely on dimensional reduction [7, 6]. Usually the assumptions of dimensional reduction, either direct or indirect, appear to be made *ad-hoc*, in order to allow the perturbative renormalizability of quantum gravity. However, several of these approaches are supported by the very geometric properties of the singularities. In perturbative expansions in terms of point-like particles, particles become tiny black holes, and the singularities weight out the amplitudes in such a way that dimensional reduction effects are possible [36].

I will now discuss the implications on what mathematical and physical objects are more fundamental for spacetime.

4 The mathematical structure of spacetime

In general relativity, spacetime is considered to be a differentiable manifold, endowed with a Lorentzian metric. This assumes implicitly an entire hierarchy of mathematical structures, and it is not easy to answer the question which are the most fundamental.

Think for example at the Euclidean plane. One may consider that the notions of distance between two points and of angle between two lines are fundamental. On the other hand, we can have already a well defined mathematical structure even without the notions of distance and angle, based only on the axioms specifying the relations between points and lines. The *affine geometry* of the Euclidean space thus relies only on the notions of points and lines, without appealing to a metric. This allows one for example to see that the affine structure of a four-dimensional Euclidean space is identical to that of the Minkowski spacetime. The difference between them is introduced by the metric. The metric notions such as distance and angle are introduced by the *congruence axioms* [16]. However, the fact that there is a rich structure already which does not rely on the metric does not prove that the Euclidean metric is less fundamental than the lines. We can proceed in a different order, and start with a *metric space*, which is a set of points endowed with a distance between pairs of points. From this, we can define a topology, and the geodesics as those continuous curves of minimal (or extremal, in general) length. Sometimes it is easier to consider the affine structure as more fundamental, while other times the metric.

More generally, from mathematical point of view, a differentiable manifold endowed with a metric, in particular the spacetime, is also a hierarchy of mathematical structures. First, the spacetime consists of events, which form a set. The set is endowed with a topology. Then, each open set of the spacetime is required to be *homeomorphic* (topologically equivalent) with an open set of an Euclidean space, this endowing the spacetime with a dimension and a structure of *topological manifold*. Since the fields satisfy equations with partial derivatives, one needs to add a *differential structure* on top of this.

Usually, the metric tensor, which gives the geometry, is introduced as the next structural level. Then, with the help of the metric, one defines a *Levi-Civita connection*, which has the role of describing the parallel transport, and from which one derives the geodesics and the Riemann curvature tensor. Then, the Einstein equation is an equation relating the curvature with the distribution of matter fields.

But geometers know that a connection does not necessarily require a metric, and it can still be used to define geodesics and a curvature. Moreover, geodesics can simply be a collection of lines, with no reference to differentiability, connection, or even length.

The metric of a general relativistic spacetime is defined at each spacetime event by 10 parameters. The metric at each event can be recovered only from the structure of lightcones, or from the null geodesics, up to a scaling factor [20]. This holds for *distinguishing spacetimes*, which cover the physically reasonable spacetimes (for instance the condition to be distinguishing rules out the closed timelike curves). Thus, the *horismos relation* (two events are in the horismos relation if and only if there is a null geodesic joining them) seems to be more fundamental than the metric, although we normally define this relations using the metric. But the fact that we can just start from a set endowed with a generic reflexive relation, which is considered to be the horismos relation, and recover most properties of a spacetime, shows that it is indeed possible that this relation is more fundamental than the metric [42]. This works for any reflexive relation, and the spacetime can even be discrete. Thus, the structure of lightcones, or the causal structure of spacetime, may be more fundamental than the metric. This plays an important role in the interpretation of the singularities that follows from the approach discussed here.

5 What mathematical formulation is more fundamental for spacetime?

If Nature prefers to use the proposed variables and atlases, it has to do this not just as a trick to avoid the infinities at singularities, but for more fundamental geometric and physical reasons. In the following, I try to elucidate these reasons.

Spacetime has a *topological*, a *differential*, and a *(geo)metric structure*, built one in top of another. The more fundamental are considered to be the topological and the differential structures. The metric is a dynamical quantity, which depends on the stress-energy of matter. Being dynamical, there is nothing to stop it from becoming degenerate at some places, and this is why singularities appear. The fact that the metric is less fundamental than the manifold structure agrees with our mathematical understanding of differential geometry. However, physically, it is possible that the *causal structure* (representing the type of intervals separating the spacetime events) is more fundamental than the differential structure. This view is supported by the fact that the topology of lightcones is not affected at the important big bang and black hole singularities, while their differential structure is affected [39]. Thus, the structure of lightcones, or the causal structure of spacetime, may be more fundamental than both the metric and the differential structure. In [39] I showed that the topology of lightcones remains intact at the known black hole and big bang singularities, and only their differential and metric structures differ from those of a Minkowski lightcone.

Another question is related to the connection and the curvature. The connection specifies isometries between the tangent spaces at infinitesimally closed events. If the lower connection is more fundamental, it should also admit a geometric interpretation. The lower connection, rather than connecting the tangent spaces, connects the tangent space at an event with the cotangent space at an infinitesimally closed event in spacetime. Its non-commutativity is expressed by the lower Riemann curvature R_{abcd}, which may be more fundamental, if we think that this tensor and not $R^a{}_{bcd}$ exhibits the known symmetries at permutations of indices, the decomposition in the Weyl and Ricci curvatures, and the corresponding spinorial decomposition.

6 What physical objects are more fundamental for spacetime?

Regarding the physical content, the proposed replacement of Einstein's equation is

$$R_{ab}\mathrm{d}_{vol} - \frac{1}{2}g_{ab}R\mathrm{d}_{vol} + g_{ab}\Lambda\mathrm{d}_{vol} = \frac{8\pi G}{c^4}T_{ab}\mathrm{d}_{vol}, \tag{1}$$

which is clearly equivalent to Einstein's outside the singularities, where $\mathrm{d}_{vol} \neq 0$, but its terms remain finite at singularities at least in some important cases. Is $T_{ab}\mathrm{d}_{vol}$ more fundamental than T_{ab}? It should be, considering that what we integrate in order to obtain the mass or the momentum are the volume forms of the form $T_{ab}u^a u^b \mathrm{d}_{vol}$. This is clear for example if the stress-energy corresponds to a fluid, $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$. One integrates the differential forms $\rho\mathrm{d}_{vol}$ and $p\mathrm{d}_{vol}$, and not the quantities ρ and p, which are not invariant, depending on the coordinates. While physicists think of them as scalar quantities, they are as scalar as the coordinates, and are defined in terms of a particular coordinate system. The truly invariant quantities are the differential forms $\rho\mathrm{d}_{vol}$ and $p\mathrm{d}_{vol}$. This is consistent with the fact that on differentiable manifolds mathematicians integrate volume forms, not scalar or tensor fields. In addition, the Lagrangian density is $R\mathrm{d}_{vol}$, and the corresponding equation is (1) rather than the usual Einstein equation, which is obtained by dividing (1) by d_{vol}, operation prohibited when the metric is degenerate, because $\mathrm{d}_{vol} = 0$, and it leads to infinities. In the particular case of the FLRW spacetime, the quantities $\rho\mathrm{d}_{vol}$ and $p\mathrm{d}_{vol}$ remain finite in the Friedman equations [41].

The above considerations suggest that the quantities used in rephrasing the geometry and physics to work at singularities are at least as adequate as the standard ones, both from physical and from geometric points of view.

7 Could spacetime be emergent?

The discussion we had so far about which mathematical and physical structures are more fundamental for spacetime assumes that spacetime is a continuum, as we understand it from general relativity. This does not exclude, however, the possibility that spacetime is discrete. There are some indications that point towards the idea that at least the information in a spacetime region has an upper bound, given by the Bekenstein-Hawking bound [5, 4]. Another argument goes along the line that since quantum mechanical systems are quantized, spacetime must be quantized as well. While this is true, the stronger argument that quantized means discrete, and spacetime therefore has to be discrete, is too far-fetched. It probably has to be true that spacetime itself is quantized in one form or another, but the argument itself is not rigorous, because even the quantum states of the Hydrogen atom are discrete only in what concerns the energy – the wavefunction of the electron in the bounded states spreads continuously throughout the entire spacetime.

In other words, when one says that a quantum system is discrete, one has to specify in which domain we consider the spectrum. When we say that the Hydrogen atom has discrete energy levels, we talk about the *frequency domain*, hence of the energy spectrum. This has no implications of discreteness on the position domain. By contrary, in the *position domain* there is no shred of discreteness, the wavefunctions are not even localized in a finite region of space. This is made clear by the uncertainty principle.

As an analogy, consider the two main types of digital graphic formats. One way is to store images as pixels, and the other way, as vector graphics. Both are constrained physically to store only a finite amount of information, but one of them is discrete in space, and the other one it is not. The vector graphics formats have infinite resolution, and the image at each possible scale is calculated using geometric formulae of lines, splines, and other geometric figures. So digital does not necessarily mean discrete in space. Similarly, spacetime can be such that the information enclosed in a finite region may be finite, without the spacetime itself being discrete.

But nevertheless there are promising approaches to quantum gravity in which spacetime itself is discrete, such as *Causal Sets* [27, 28], *Causal Dynamical Triangulations* [1, 2], *Loop Quantum Gravity* (LQG) [24], *emergent gravity* [45, 44] *etc.* The possibility that spacetime itself is discrete is currently under active consideration, perhaps more than ever.

If spacetime itself is discrete, it may seem easier to impose conditions that avoid the singularities. For example, the Einstein equations are not really a limit of those of LQG, but merely an approximation. For example, in *Loop Quantum Cosmology* it is easy to impose conditions that remove the big bang singularity [3]. These conditions are not compatible with the hypothesis of the singularity theorems in general relativity, and this is why it is possible to avoid the occurrence of singularities.

However, even if it is the case that spacetime itself is discrete, and it might very well be, the results presented here about the structure of singularities can turn out to be useful. It may still be possible to have degenerate metrics, and hence singularities of this kind. Or, if we remove them by some condition in some discrete spacetime theory of quantum gravity, at least the singularities will correspond in that theory to situations where the arrangement of the spacetime "atoms" is special, in the sense that it is highly degenerate or extreme in some directions. For instance, the big bounce from Loop Quantum Cosmology has a "bottleneck" of highest curvature and smallest radius, where the bounce happens.

The experimental tests of general relativity show that the theory, even if it would not be completely accurate, is at least a surprisingly good approximation of reality. This is usually explained by assuming that the discreteness is manifest at very small scales. But there are other reasons that ensure this. In [42] I showed that, no matter whether the spacetime is discrete or not, or maybe a hybrid between the two, one can derive properties of spacetime, like a topology, geodesics, dimension, and the metric up to a scaling factor, from the horismos relations alone, with minimal additional constraints mainly in the case of dimension. So the conceptual content of general relativity is something that remains stable and goes far beyond the continuous-discrete dichotomy. And the fact that the horismos relation, or the causal structure, are so fundamental, is also a lesson we learn from the approach I proposed to spacetime singularities [39].

8 Conclusions

I explained that it is possible to formulate the equations of general relativity in a way which is equivalent to the standard way outside the singularities, but in addition can be applied at the singularities, where they yield finite quantities. The geometric and physical objects that I used in this approach are invariant and natural. The fact that the proposed formulation appears to work in regimes in which the usual formulation does not work raises questions like: "Are there other reasons to accept the proposed formulation? The standard formulation of general relativity appeared to be natural both from differential-geometric and physical reasons. What if the alternative formulation proposed here is unnatural?".

Such questions are legitimate, and I addressed them in this article. We have seen that even for the Euclidean geometry of the plane, the mathematician does not always have a reason to consider one of the structures more fundamental than another. The choice is usually dictated by the interest in a particular structure, and by the applications. Mathematicians can take any of the structures in the hierarchy and abstract them. The notion of *forgetful functor* from *category theory* [19] allows ones to move from the category of mathematical structures of a kind to the category of mathematical structures of other kind, while ignoring or forgetting some of the mathematical structures. This abstraction does not have a prescribed order of forgetting, but rather there are more paths, and the different orderings of abstractions commute. So how do we know which of the mathematical structures is more fundamental in general relativity?

It appears natural to consider the most general structures as being more fundamental. With this choice, the lower covariant derivative and the lower Riemannian curvature appear to be more fundamental, since they apply to more situations. The usual covariant derivative and Riemann curvature obtained from a non-degenerate metric make sense only for such metrics, while the ones from the proposed approach make sense to more general metrics, which include the degenerate case and the main known singularities in general relativity.

From physical point of view, as seen from the example of the FLRW spacetime, the densitized formulation is more natural, since the energy and pressure densities $\rho\mathrm{d}_{vol}$ and $p\mathrm{d}_{vol}$ correspond to actual densities, while their scalar counterparts ρ and p are dependent on the coordinates in which $\det g_{ab} = 1$, and are not densities, being scalars. When performing integration, the densities are the actual integrand, while in order to integrate scalars, one needs to add the volume form as a correction for the integral to make sense, as it is known from any textbook of analysis on manifolds [21].

However, while these arguments allow us to select the more adequate geometric and physical quantities, they have no implication on the choice of the prefered atlas or differential structure. In this case, in the spirit of what Eddington and Finkelstein did, I suggest it is preferable to choose the atlas which allows the quantities to remain finite. I think this argument leaves room for improvement even in their case, because we do not have a mathematical result which shows that this choice is unique. It may very well be possible that a different choice leads to finite solutions, yet the solutions, when extended beyond the singularity, may be different. In the case of the Schwarzschild solution [31], an infinity of possible singular coordinate transformations are available to make the Schwarzschild metric degenerate. However, only one of the transformations I found results in a semi-regular metric in the sense of [37]. This hints towards the possible existence of a criterion for selecting the fundamental atlas, which still remains to be found.

A relevant hint in the direction of finding this criterion follows from the importance of the causal structure of spacetime, in particular of the horismos relation. The lightcone structure is strongly distorted at singularities, but since the lightcones preserve at least their topological structure, this indicates that they may be more fundamental not only than the metric, but even than the differential structure, as explained in [42, 39]. So the condition that the topology of the lightcones remains intact at singularities may be a relevant condition which allows us to select the correct atlas in which the metric tensor to be made benign or, if possible, non-singular, even if in the original atlas it appeared to be malign.

These discussions suggest that it is time to reconsider the ontology of spacetime in general relativity, and we accumulated several new tools that allow us to do this in a deeper conceptual way. Spacetime has still much to teach us, and its nature and ontology should always remain an object of critical investigation, as it is relevant not only for the philosophical foundations, but also as guideline for the new theories of quantum gravity.

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