Thermodynamic manifolds and stability of black holes in various dimensions

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“Second Hermann Minkowski Meeting on the Foundations of Spacetime Physics”
Albena, Bulgaria

May 14, 2019
Black hole thermodynamics

Black holes have entropy (Bekenstein-Hawking ’70s):

\[ S = k_B \frac{A}{4L_p^2} + \text{corrections} \]  \hspace{1cm} (1)

The first law of thermodynamics:

\[ dM = TdS + \Omega dJ + \Phi dQ + \cdots = TdS + \Phi_i dQ^i = I_a dE^a \] \hspace{1cm} (2)

Black holes thermal stability (Davies ’80):

\[ C = Tr \frac{\partial S}{\partial T} = \begin{cases} > 0, & \text{stable}, \\ < 0, & \text{radiating (unstable)}, \\ = 0, & \text{phase transitions}, \\ \to \infty, & \text{phase transitions} \end{cases} \] \hspace{1cm} (3)
Geometric approaches to the equilibrium space of black holes

The space of extensive parameters $\mathcal{E} = \{\Xi, E^a\}$ is called an equilibrium manifold if supplied with a proper metric structure.

- Legendre invariant metrics, “Geometrothermodynamics” (H. Quevedo 2006)
Hessian metrics

Fluctuation theory (G. Ruppeiner ‘79):

\[ S(E^a) = S_0 + EQL + \frac{\partial^2 S}{\partial E^a \partial E^b} dE^a dE^b + \cdots \]

\[ = S_0 + EQL - g_{ab}(\vec{E}) dE^a dE^b \]  \hspace{1cm} (4)

Ruppeiner information metric:

\[ g^{(R)}_{ab} = - \frac{\partial^2 S}{\partial E^a \partial E^b} = -\text{Hess} S(\vec{E}) \]  \hspace{1cm} (5)

Weinhold information metric (F. Weinhold ‘75):

\[ g^{(W)}_{ab} = \frac{\partial^2 M}{\partial E^a \partial E^b} = \text{Hess} M(\vec{E}) \]  \hspace{1cm} (6)
The probability for fluctuating between two macro states is proportional to the geodesic distance between them in $\mathcal{E}$. 

The strength of interactions/correlations between quantum bits on the event horizon = the magnitude of $R$.

The sign of $R$ indicates the type of interactions (G. Ruppeiner '10):

$$ R \begin{cases} > 0, & \text{repulsive interactions,} \\ < 0, & \text{attractive interactions,} \\ = 0, & \text{free theory,} \\ \to \infty, & \text{phase transitions} \end{cases} \quad (7) $$

Phase transitions = divergencies of $R$ (F. Weinhold '75, G. Ruppeiner '79)
• Consider \((2n + 1)\) TD phase space \(\mathcal{T}\) with coordinates 
\[ Z^A = (\Xi, I^a, E^a), \quad a = 1, \ldots, n, \] 
where \(\Xi\) is a TD potential.

• Select on \(\mathcal{T}\) a non-degenerate Legendre invariant metric 
\[ G = G(Z^A) \] 
and a Gibbs 1-form \(\Theta(Z^A)\), namely

\[
G^{GTD} = \Theta^2 + (\xi_{ab} E^a I^b)(\eta_{cd} dE^c dI^d), \quad \Theta = d\Xi - \delta_{ab} I^a dE^b,
\]
where \(\delta_{ab}\) is the identity matrix, \(\eta_{ab}\) is the Minkowski metric, 
and \(\xi_{ab}\) is some constant tensor.

• Take the pullback \(\phi^* : \mathcal{T} \rightarrow \mathcal{E}\) to find (H. Quevedo ’17):

\[
ds^2 = \left( \delta_{ac} \xi^{cb} E^a \frac{\partial \Xi}{\partial E^b} \right) \left( \eta^d \frac{\partial^2 \Xi}{\partial E^d \partial E^f} dE^e dE^f \right)
\]

\[ (8) \]
Conjugate thermodynamic potentials

For general black holes with \((m + 1)\) TD variables, \((S, \Phi_i)\), and Enthalpy potential, \(\bar{M} = M - \Phi_i Q_i\), one can define the metric (B. Mirza, A. Mansoori ’19):

\[
\hat{g} = \text{blockdiag} \left( \frac{1}{T} \frac{\partial^2 M}{\partial S^2}, -\hat{G} \right),
\]

(9)

where

\[
G_{ij} = \frac{1}{T} \frac{\partial^2 M}{\partial Y^i \partial Y^j}, \quad Y^i = (Q_1, \ldots, Q_m)
\]

(10)
Black holes in 3 and 4 dimensions

1. TIG for 4d Deser-Sarioglu-Tekin black hole solution in higher derivative theory of gravity (T. Vetsov ’19):

\[ I = \frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{-g} \left( R + \sigma \sqrt{3} \text{Tr}(\hat{C}^2) \right), \quad \sigma < -\frac{1}{2} \text{ & } \sigma > 1 \]

2. TIG for 3d WAdS\(_3\) black hole solution in TMG dual to WCFT\(_2\) with left and right central charges (H. Dimov, R. C. Rashkov, T. Vetsov ’19)

\[ I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} \left( R + \frac{2}{L} \right) + \frac{1}{\mu} I_{CS} + \int_{\partial\mathcal{M}} B \]

\[ T_c = \frac{1}{\pi (c_L + \sqrt{c_L c_R})} \]
Summary

- Thermodynamic information geometry (TIG) is a set of geometric tools for investigating statistical thermal systems in equilibrium or non-equilibrium.
- TIG is a subset in the more powerful framework of Information Geometry (IG).
- IG is essential for understanding how classical and quantum information can be encoded onto the degrees of freedom of any physical system.
- Growing number of applications beyond physics.
Thank You!

- In collaboration with R. C. Rashkov and H. Dimov:
- Partially supported by
  - The Bulgarian NSF Grant DM 18/1
  - The Sofia University Grant 10-80-149