Universality of the Einstein theory of gravitation

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We show that generalizations of general relativity theory, which consist in replacing the (linear in scalar curvature) Hilbert Lagrangian $L_{Hilbert} = \frac{1}{16\pi}\sqrt{|g|}R$, by a generic scalar density $L = L(g_{\mu\nu}, R^{\lambda}_{\mu\nu\kappa})$ depending upon the metric $g_{\mu\nu}$ and the whole curvature tensor $R^{\lambda}_{\mu\nu\kappa}$, are equivalent to the conventional Einstein theory for a (possibly) different metric tensor $\tilde{g}_{\mu\nu}$ and (possibly) a different set of matter fields.

More precisely, consider a ,,generalized" theory of gravity, based on an invariant Lagrangian density:

$$L = L(g_{\mu\nu}, R^{\lambda}_{\mu\nu\kappa}, \Gamma^{\lambda}_{\mu\nu}, \varphi, \partial\varphi) \ , \tag{1}$$

where $\Gamma^{\lambda}_{\mu\nu}$ is a Levi-Civita connection of the metric $g_{\mu\nu}$ and $R^{\lambda}_{\mu\nu\kappa}$ denotes its Riemann tensor, whereas φ denotes some matter fields. The following mathematical statement can be proved:
Theorem 1: There exists a one-to-one change of variables:

$$(g, \varphi) \Longleftrightarrow (\tilde{g}, \varphi, \phi) \ , \tag{2}$$

and a new matter Lagrangian:

$$\tilde{L}_{Matter} = \tilde{L}_{Matter}(\tilde{g}, \partial\tilde{g}, \varphi, \phi, \partial\varphi, \partial\phi) \ , \tag{3}$$

such that (g, φ) satisfy field equations derived from the Lagrangian (1) if and only if the corresponding fields $(\tilde{g}, \varphi, \phi)$ satisfy the conventional ,,Einstein + matter" equations, derived from the conventional Hilbert variational principle:

$$\tilde{L} := L_{Hilbert}(\tilde{g}) + \tilde{L}_{Matter} \ . \tag{4}$$

In particular, equations for the new metric $\tilde{g}$ are of the second differential order: $G^{\mu\nu}(\tilde{g}) = 8\pi\tilde{T}^{\mu\nu}$, with

$$\tilde{T}^{\mu\nu} := \frac{2}{\sqrt{|\tilde{g}|}}\frac{\delta\tilde{L}_{Matter}}{\delta\tilde{g}_{\mu\nu}} \ . \tag{5}$$

whereas equations for the old metric g, derived from the original Lagrangian density (1), were of the fourth order. Also matter field equations are of the second differential order because $\tilde{L}_{Matter}$ depends upon first derivatives only.

Recently, we have been able to extend our result to higher order Lagrangians (i.e. admitting covariant derivatives of the Riemann tensor up to order k).

The particular case of a Lagrangian L which depends non-linearly upon the Ricci tensor, but does not depend upon the Weyl tensor, were first considered by Stephenson and Higgs (see [1]) and later analyzed thoroughly by many authors (cf. [2]). Equivalence of such theories with the standard General Relativity Theory was proved already long ago (see e.g. [3]). In particular, the Sacharov's non-linear Lagrangian containing the R^2 term (see [4]) is equivalent to the standard GR interacting with a scalar field (see [5] and also [3]).

But there is a renewed interest in such (or even more radical) generalizations of the Einsteinian theory of gravity (see e.g. [6]). In this context our result can be summarized as follows: generalizations of the theory of gravity, based on non-conventional Lagrangians are equivalent to the conventional (i.e. Einsteinian) theory of gravity, interacting with non-conventional matter fields. Einstein theory is, therefore, universal!

References

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