# Premetric gravity

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*The premetric formalism* is an alternative representation of a classical field theory in which the field equations are formulated without the spacetime metric. Only the constitutive relations between the basic field variables, excitation $H$ and field strenghth $F$, can involve the metric of the underlying manifold. This idea can be traced back to the early 1920s where it appears in the publications of Kottler. Various applications of this construction to the formal structure of electrodynamics were worked out by Post [1]. The premetric formalism was studied intensively in the book [2]. For an account of the recent developments in this area, see our review [3].

One advantage of the premetric formalism is that the validity of a certain premetric model can be extended to a more general spacetime geometry.

The premetric construction works pretty well in Maxwell's classical electrodynamics. In this case, all basic ingredients, such as the field equations, the conserved quantities of electric charge and of magnetic flux, and the Lorentz force expression are presented in a metric-free form. Only the constitutive relation between the excitation and the field strength are formulated with the use of the metric tensor.

Although Kottler's premetric program works well in Newtonian gravity and even in a flat gravitomagnetism model [3], it seems to be completely unacceptable in general relativity (GR). This is due to the well-known fact that Einstein's theory is essentially based on a pseudo-Riemann geometry with the metric tensor as its primary variable. Nevertheless, in this paper, we will show that a premetric construction of GR is possible if one turns to its teleparallel reformulation. We construct a teleparallel model for the coframe field. It is a vector-valued analog of electromagnetic theory with a well-defined gravitational energy-momentum current and a Lorentz-type force density. The analogy between our premetric coframe model of gravity and the standard electromagnetic theory is underlined in Table 1.

We consider the coframe model on a pseudo-Riemannian manifold. The general local linear constitutive law between the coframe excitation and the coframe field strength is defined by the use of a constitutive pseudotensor of 6th rank. We turn now to the gravitational model. We require the 6th rank constitutive tensor to be expressed in terms of the metric tensor $g_{\alpha\beta}$ as variable alone. The most general expression of this type appears to be

$$\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu} = \frac{\sqrt{-g}}{\varkappa}\Big\{\beta_1 g_{\alpha\mu} g^{\nu[\beta} g^{\gamma]\rho} + \beta_2 \left(g^{\rho[\gamma}\delta^{\beta]}_{\alpha}\delta^{\nu}_{\mu} - g^{\nu[\gamma}\delta^{\beta]}_{\alpha}\delta^{\rho}_{\mu}\right) + \beta_3 \left(g^{\rho[\gamma}\delta^{\beta]}_{\mu}\delta^{\nu}_{\alpha} - g^{\nu[\gamma}\delta^{\beta]}_{\mu}\delta^{\rho}_{\alpha}\right)\Big\}\,, \tag{1}$$

The natural requirement of the localization of the group of coframe transformations yields the free parameters to be

$$\beta_1 = 1\,, \beta_2 = -4\,, \beta_3 = 2\,. \tag{2}$$

This model turns out to be fully equivalent to GR.

We construct a metric-free equation for a congruence of trajectories with a constitutive law between the momentum covector and the velocity vector. Its restriction to the metric manifold yields a geodesic curve in the gravitational case and a trajectory of a charge in an exterior field in the electromagnetic case.

**Table I. Premetric electromagnetism-gravity analogy.**

| Objects and Laws | Electromagnetism | Gravity |
|---|---|---|
| Source current | $J$ | $\Sigma_\alpha$ |
| Conserved source current | $dJ = 0$ | $d\Sigma_\alpha = 0$ |
| Excitation | $H$ | $H_\alpha$ |
| Inhom. field equation | $dH = J$ | $dH_\alpha = {}^{(\vartheta)}\Sigma_\alpha + {}^{(\mathrm{m})}\Sigma_\alpha$ |
| Field strength | $F$ | $F^\alpha$ |
| Hom. field equation | $dF = 0$ | $dF^\alpha = 0$ |
| Potential | $A$ | $\vartheta^\alpha$ |
| Potential equation | $dA = F$ | $d\vartheta^\alpha = F^\alpha$ |
| Lorentz force | $f_\alpha = (e_\alpha \rfloor F) \wedge J$ | $f_\alpha = \left(e_\alpha \rfloor F^\beta\right) \wedge {}^{(\mathrm{m})}\Sigma_\beta$ |
| Energy-momentum | $\Sigma_\alpha = e_\alpha \rfloor \Lambda + F \wedge (e_\alpha \rfloor H)$ | ${}^{(\vartheta)}\Sigma_\alpha = e_\alpha \rfloor \Lambda + F^\beta \wedge (e_\alpha \rfloor H_\beta)$ |
| Lagrangian | $\Lambda = -(1/2) F \wedge H$ | $\Lambda = -(1/2) F^\alpha \wedge H_\alpha$ |
| Constitutive tensor | $\chi^{\alpha\beta\gamma\delta}$ | $\chi^{\beta\gamma}{}_{\alpha}{}^{\nu\rho}{}_{\mu}$ |

# References


[1] E. J. Post, *Formal Structure of Electromagnetics – General Covariance and Electromagnetics* (North Holland, Amsterdam, 1962, and Dover, Mineola, NY, 1997).

[2] F. W. Hehl and Yu. N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser, Boston, MA, 2003).

[3] F. W. Hehl, Y. Itin and Y. N. Obukhov, On Kottler's path: origin and evolution of the premetric program in gravity and in electrodynamics, arXiv:1607.06159; a condensed version appeared in the Int. J. Mod. Phys. D **25**, 1640016 (2016).

[4] Y. Itin, F. W. Hehl and Y. N. Obukhov, "The premetric equivalent of general relativity: teleparallelism," arXiv:1611.05759 [gr-qc].