Branched Covering Spaces and Partition Functions in Quantum Gravity

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Differential geometry and topology are fundamentally important to the analysis of spacetime models. Not just because the physical theory which describes spacetime is built upon them, but they are also naturally associated with the physical concept of space. Like so many close connections in theoretical physics and mathematics, interesting features and results from one inform the explorations of the other. A specific example of this is that of the classification problem of smooth 4-manifolds, in which the key issue is exotic smooth structure. Attempts to understand and model exotic smooth structure have lead to new threads of exploration, such as the paradigm of using branched covering spaces and partition functions to study quantum gravity.

The nature of exotic smooth structure can be understood through a careful analysis of the construction of a smooth manifold. When passing from a topological manifold to a smooth one, one must select a set of smooth transition functions between coordinate patches, upon which the higher level geometric structures will be built - tangent spaces, bundles, connections, *etc.* It has been known for some time (on 7-spheres [13]) that the choice of transition functions may not be unique, but general results took until the 1980s to develop [18, 11, 6]. With the proof of the geometrization conjecture in 2003 [14], the smooth classification problem is open only in dimension 4. However, it is known that in many cases closed 4 manifolds can have an infinite number of exotic smooth structures, and of particular interest is that this is also true in the most common model for spacetime, $\mathbb{R}^4$ [19].

Although tools from gauge theory were used in the work on 4-manifold topology, proposals for the influence of exotic smooth structure on physical models did not start to appear until the 1990s. First introduced in [5], it was pointed out that a region of spacetime could be described by localized exotic smooth structure, and would appear to be a non-luminious gravitational source. The proposal that dark matter can be ascribed to exotic smooth structure (rather than exotic physical interactions) is generally referred to as The Brans Conjecture [3]. In light of the failure of the LHC to find direct evidence for dark matter, this model has recently been receiving more attention [1].

The techniques required to study the impact of exotic smooth structure on physical models have been developing over the past 20 years. First it was demonstrated that for a specific case in dimension 7, the expectation value of volume was different if exotic smooth structures were included [17]. This approach used a semiclassical partition function, so the result was tantamount to finding metrics that were nondiffeomorphic to the standard one (and to each other), so they represented inequivalent solutions to the Einstein equations. Using this same approach, the author of this abstract was able to demonstrate a similar phenomena in dimension 4 [7]. In this case, the manifolds were presented as branched covering spaces, which allowed for the easy calculation of the required Seiberg-Witten invariants [16].

It was soon recognized that the branched cover construction above is a specific case of a more general mathematical result - any 4-manifold can be presented as a cover of a 4-sphere branched over a surface [15]. In addition, these surfaces can be specified explicitly via spinors by using a Weierstrass representation [8]. In fact, the Weierstrass representation of surfaces in three dimensions has also been shown to be useful to study exotic smooth structure. For instance, such surfaces can reproduce matter terms in the Einstein-Hilbert action, another step towards proving the Brans Conjecture [4, 2].

In the current work we explore more deeply the presentation of 4-manifolds as branched covers of the 4-sphere, utilizing semiclassical partition functions as the basic calculational tool. It is demonstrated that because the order of the cover is bounded ([12, 8]), the action on each cover only differs by the order of the covering over the ramification locus (that is, the surfaces). The value of the action can then be tracked by an explicit presentation of the fundamental group [10].

The degrees of freedom can be further clarified by representing the surfaces as flat with conical singularities. With the topology and geometry now both explicit, the classification problem is circumvented, although at the expense of an unfamiliar representation of the set of gravitational instantons. Since any smooth 4-manifold can be presented in this way, the partition function is formally complete. In addition, with a close connection between the topology of the surfaces and the topology of the covers, it should be possible to study the topology of spacetime using observations of the early universe cosmic string density [9].

The previous statements are based on tree-level calculations using the partition function, and due to the fact that the order of the cover is bounded, the covers are isovolumetric. Preliminary work on the 1-loop calculations indicate that this symmetry can be violated, and because of the conical singularities, regularization-dependent terms are introduced into the action. At this stage, it is not yet clear how to handle these additional terms.

The original discovery of exotic 7-spheres (as well as the later 4-dimensional revolution) had a major impact on the field of differential geometry. Using modern techniques and alternative representations, it is now possible to study these features in theoretical physics, and determine what role they may play in our spacetime model. This is not only a question of mathematical rigor, but also a critical question about the validity of our extragalactic and cosmological models, upon which so much of our understanding of the universe is based.

References

[1] T. Asselmeyer-Maluga. Smooth quantum gravity: Exotic smoothness and quantum gravity. In *At the Frontier of Spacetime*, Fundamental Theories of Physics. Springer International Publishing, 2016.

[2] T. Asselmeyer-Maluga and C. Brans. Smoothly exotic black holes. In *Black Holes: Evolution, Theory and Thermodynamics*, Space Science, Exploration and Policies, Physics Research and Technology. NOVA Science Publishers, 2012.

[3] T. Asselmeyer-Maluga and C. H. Brans. *Exotic smoothness and physics.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[4] T. Asselmeyer-Maluga and H. Rosé. On the geometrization of matter by exotic smoothness. *Gen. Relativity Gravitation*, 44(11):2825–2856, 2012.

[5] C. H. Brans. Localized exotic smoothness. *Classical Quant. Grav.*, 11(7):1785–1792, 1994.

[6] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[7] C. L. Duston. Exotic smoothness in four dimensions and Euclidean quantum gravity. *Int. J. Geom. Methods Mod. Phys.*, 8(3):459–484, 2011.

[8] C. L. Duston. Semiclassical partition functions for gravity with cosmic strings. *Classical and Quantum Gravity*, 30(16):165009, Aug. 2013.

[9] C. L. Duston. Using Cosmic Strings to Relate Local Geometry to Spatial Topology. *ArXiv e-prints*, June 2015.

[10] R. H. Fox. A quick trip through knot theory. In *Topology of 3-Manifolds and related topics*, pages 120–167. Prentice-Hall, 1962.

[11] M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.

[12] M. Iori and R. Piergallini. 4-manifolds as covers of the 4-sphere branched over non-singular surfaces. *Geom. Topol.*, 6:393–401, 2002.

[13] J. W. Milnor. *Topology from the differentiable viewpoint.* Based on notes by David W. Weaver. The University Press of Virginia, Charlottesville, Va., 1965.

[14] J. W. Morgan. Recent progress on the Poincaré conjecture and the classification of 3-manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 42(1):57–78, 2005.

[15] R. Piergallini. Four-manifolds as 4-fold branched covers of S^4. *Topology*, 34(3):497–508, 1995.

[16] M. Salvetti. On the number of nonequivalent differentiable structures on 4-manifolds. *Manuscripta Math.*, 63(2):157–171, 1989.

[17] K. Schleich and D. Witt. Exotic spaces in quantum gravity. I. Euclidean quantum gravity in seven dimensions. *Classical Quant. Grav.*, 16(7):2447–2469, 1999.

[18] S. Smale. On the structure of manifolds. *Amer. J. Math.*, 84:387–399, 1962.

[19] C. H. Taubes. Gauge theory on asymptotically periodic 4-manifolds. *J. Differential Geom.*, 25(3):363–430, 1987.