Is Spacetime as Physical as Is Space?

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Newton’s construction of mechanics started from separated space and time. This can be inserted into the spacetime formalism by considering a spacetime manifold that is a product: $\mathrm{V}_{\mathrm{N-L}} = \mathrm{A}^1 \times \mathrm{A}^3$, with the time axis A^1 being a one-dimensional affine space and with the space being a three-dimensional affine space A^3. That product spacetime manifold $\mathrm{V}_{\mathrm{N-L}}$ obviously admits privileged “space” and “time” projections. Thus, it admits a preferred reference space (Newton’s “absolute space”) and a preferred reference time (the “absolute time”). This has been appropriately called the “Newton-Lorentz Universe” [1]: the additional reference to Lorentz is there to indicate that indeed the Galilean relativity does not necessarily apply in such an universe. (In its first formulation, Lorentz’s electromagnetic theory admitted preferred-frame effects, though actually not at the first order in v/c.) But the Galilean relativity does of course apply to Newton’s mechanics and it implies that there is no physically preferred *space.* This is compatible with the product manifold $\mathrm{V}_{\mathrm{N-L}}$ because, beyond the mathematical structure of spacetime, there is still the dynamics in Newton’s mechanics.

Yet Galilean relativity can be made apparent in the structure of the spacetime, if instead of defining it as a product one takes it as a 4-D affine space A^4 whose *translation space* E^4 (a 4-D vector space) is endowed with a preferred “time” map $t : \mathrm{E}^4 \to \mathbb{R}$; the time interval between two events $a, b \in \mathrm{A}^4$ is $\delta t(a, b) = t(b - a)$ [2]. The product structure remains true in a weaker sense for this “Galileo Universe” considered by Arnold [2]: since $\mathrm{A}^1 \times \mathrm{A}^3$ is an affine space of dimension 4, the spacetime A^4 is isomorphic as an affine space to $\mathrm{A}^1 \times \mathrm{A}^3$. However, there is no *canonical* (preferred) isomorphism of A^4 onto $\mathrm{A}^1 \times \mathrm{A}^3$. In addition to the affine space structure, in both the Galileo Universe of Ref. [2] and the Newton-Lorentz Universe there is also an Euclidean spatial metric. Thus the Newton-Lorentz Universe is physically more general than the Galileo Universe — since in the former the Galilean relativity does not necessarily apply, though it does apply if one defines the dynamics as Newton did. One might say, however, that the former is mathematically less general (since in it there does exist canonical “space” and “time” projections) than is the latter. In any case, that discussion shows that, for classical mechanics, the correspondence between the physics and the mathematical structure of spacetime is not one-to-one.

It is often considered that the only spacetime which is relevant to special relativity is the Minkowski spacetime $(\mathrm{A}^4, \boldsymbol{\gamma})$. Here A^4 is indeed the same 4-D affine space as for the Galilean spacetime, that is isomorphic as an affine space to $\mathrm{A}^1 \times \mathrm{A}^3$ but not canonically so, and $\boldsymbol{\gamma}$ is the (Poincaré-)Minkowski metric: a flat Lorentzian metric "on A^4 ", acting in fact on pairs of vectors in the translation space E^4. However, just as in the case of classical mechanics, we may also start from the Newton-Lorentz Universe $\mathrm{V}_{\mathrm{N-L}} = \mathrm{A}^1 \times \mathrm{A}^3$. Then we may endow the respective translation spaces E^1 and E^3 with Euclidean metrics, say $\boldsymbol{h}^1$ and $\boldsymbol{h}^3$, and from these define the Minkowski metric on the translation space of $\mathrm{A}^1 \times \mathrm{A}^3$, that is $\mathrm{E}^1 \times \mathrm{E}^3$:

$$\boldsymbol{\gamma}(V, V') = \boldsymbol{\gamma}((\tau, v), (\tau', v')) := \boldsymbol{h}^1(\tau, \tau') - \boldsymbol{h}^3(v, v'). \tag{1}$$

This is a way of formalizing the Lorentz-Poincaré version of special relativity, which starts from absolute time and space and ends up with Minkowski metric on spacetime. Thus either the Galilean relativity or the relativity of Poincaré and Einstein can be implemented on a common preexisting structure with preferred time and preferred space, $\mathrm{V}_{\mathrm{N-L}} = \mathrm{A}^1 \times \mathrm{A}^3$.

In contrast, the spacetimes relevant to general relativity are general Lorentzian 4-D manifolds $(\mathrm{V}, \boldsymbol{g})$ (more exactly, those which are such that their Lorentzian metric $\boldsymbol{g}$ is a solution of the Einstein equations), for which the manifold V is not in general diffeomorphic to a product manifold and is much less often an affine space. With such general spacetime manifolds, arises the question of how to adequately define the space, which is obviously needed for concrete physical problems. This is related with the notion of a reference fluid [3]. In general both a reference fluid and an associated 3-D space manifold can be defined from the data of a non-vanishing vector field v on V: the tangent vector field to some 3-D congruence of world lines or "observers" — provided the flow of v behaves "normally" [4]; this does not need that there be a metric on V. The associated space manifold is the set of the world lines of the observers.

The foregoing seems to lead to the following view. On the one hand, spacetime is a very useful and clever mathematical construction, but it is not uniquely related with the physics. On the other hand, although there are many equally valid physical spaces in our physical word, each of them is uniquely defined from a congruence of observers, and at least in simple cases any two of them are isomorphic.

References

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